Neyman-Pearson Lemma

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Most Powerful Test

- Suppose we want to test the simple hypothesis $H_0 : \theta = \theta_0$ against the simple hypothesis $H_1 : \theta = \theta_1$ where θ_0 and θ_1 are specified.
- If there is a critical region ω which minimizes the power $1 - \beta$ amongst all critical regions of size α then it is said to be the best critical region.
- The test corresponding to the best critical region is called the most powerful test.

Neyman-Pearson Lemma

- The Neyman-Pearson fundamental lemma specifies the BCR or MP size α test.
- Let X be a random variable with pdf $f(x, \theta)$ where θ is unknown. Suppose we want to test the hypothesis:

$$H_0 : \theta = \theta_0$$

vs

$$H_1 : \theta = \theta_1$$

where θ_0 and θ_1 are specified.

- Let $X_1, X_2, ..., X_n$ be independent observations on X with corresponding likelihood function $L(X, \theta)$. Then the BCR of size α for testing H_0 against H_1 is given by the critical region

$$\omega = \{X : \frac{L(X, \theta_1)}{L(X, \theta_0)} > k\}$$

Example 1

Use the Neyman-Pearson lemma to obtain the region for testing $H_0 : \mu = \mu_0$ against $H_1 : \mu = \mu_1$ in the case of a normal population $N(\mu, \sigma^2)$ where σ^2 is known. Find the power of the test.

Solution

- $$f(x,\mu) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$$
- $$L(X,\mu) = \prod_{i=1}^{n} f(x_i,\mu) = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2}$$
- Under H_0

$$L(X,\mu_0) = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(X_i-\mu_0)^2}$$

Cont'd

Applying the lemma the BCR is given by

$$
\begin{aligned}
\omega &= \{X : \frac{L(X, \mu_1)}{L(X, \mu_0)} > k\} \\
&= \{X : \frac{e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(X_i - \mu_1)^2}}{e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(X_i - \mu_0)^2}} > k\} \\
&= \{X : e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(X_i - \mu_1)^2} + e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(X_i - \mu_0)^2} > k\} \\
&= \{X : -\frac{1}{2\sigma^2}\sum_{i=1}^{n}(X_i - \mu_1)^2 + -\frac{1}{2\sigma^2}\sum_{i=1}^{n}(X_i - \mu_0)^2 \geq \ln k\} \\
&= \{X : -\frac{1}{2\sigma^2}\Big[\sum_{i=1}^{n} X_i^2 - 2\mu_1 \sum_{i=1}^{n} X_i + n\mu_1^2 - \sum_{i=1}^{n} X_i^2 + 2\mu_0 \sum_{i=1}^{n} X_i - n\mu_0^2\Big] >
\end{aligned} \tag{1}
$$

Cont'd

- Collecting like terms together and simplifying

$$\begin{aligned}\omega &= \{X : -\frac{n}{2\sigma^2}(\mu_1^2 - \mu_0^2) + \frac{1}{\sigma^2}(\mu_1 - \mu_0)\sum_{i=1}^{n} X_i > \ln k\} \\ &= \{X : \frac{1}{\sigma^2}(\mu_1 - \mu_0)\sum_{i=1}^{n} X_i > \ln k + \frac{n}{2\sigma^2}(\mu_1^2 - \mu_0^2)\}\end{aligned} \tag{2}$$

- We consider two cases:
  i. $\mu_1 > \mu_0$
  ii. $\mu_1 < \mu_0$

Case I: $\mu_1 > \mu_0$

- In this case:

$$\bar{X} > \frac{\sigma^2 \ln k}{n(\mu_1 - \mu_0)} + \left(\frac{\mu_1 + \mu_0}{2}\right)$$

i.e $\bar{X} \geq C_\alpha$ and C_α satisfies

$$P[X \in \omega | H_0] = P[\bar{X} > c_\alpha | \mu = \mu_0] = \alpha$$

but

$$X \sim N(\mu, \sigma^2) \rightarrow \bar{X} \sim N(\mu, \frac{\sigma^2}{n})$$

Cont'd

- Therefore when H_0 is true then $\bar{X} \sim N(\mu_0, \frac{\sigma^2}{n})$

$$
\begin{aligned}
\alpha &= P[\bar{X} > C_\alpha | H_0] \\
&= P[\bar{X} > C_\alpha | \mu = \mu_0] \\
&= P\left[\frac{\bar{X} - \mu_0}{\frac{\sigma}{\sqrt{n}}} > \frac{C_\alpha - \mu_0}{\frac{\sigma}{\sqrt{n}}}\right] \\
&= P[z > \frac{C_\alpha - \mu_0}{\frac{\sigma}{\sqrt{n}}}] = \alpha
\end{aligned}
\tag{3}
$$

Cont'd

- $P[z > z_{1-\alpha}] = \alpha$
- $$Z_{1-\alpha} = c_\alpha - \frac{\mu_0}{\sigma/\sqrt{n}}$$
- $$C_\alpha = \frac{\sigma}{\sqrt{n}}(Z_{1-\alpha}) + \mu_0$$
- Thus the critical region is given by:

$$\omega : \bar{X} > \frac{\sigma}{\sqrt{n}}(Z_{1-\alpha}) + \mu_0$$

Case II: $\mu_1 < \mu_0$

- In this case the BCR is given as:

$$\bar{x} < \frac{\sigma^2}{n}\frac{\ln k}{\mu_1 - \mu_0} + \frac{1}{2}(\mu_1 + \mu_0)$$

  From equation 11. That is $\bar{x} < C_\alpha$

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$$\begin{aligned} P(X \in \omega | H_0) &= P[\bar{X} < C_\alpha | H_0] = \alpha \\ &= P\left[\frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} < \frac{c_\alpha - \mu_0}{\sigma/\sqrt{n}}\right] = \alpha \\ &= P\left[Z < \frac{c_\alpha - \mu_0}{\sigma/\sqrt{n}}\right] = \alpha \end{aligned} \tag{4}$$

Cont'd

- Therefore $P[z < Z_\alpha] = \alpha - Z_\alpha = \frac{c_\alpha - \mu_0}{\sigma/\sqrt{n}}$
- Thus the critical region is:

$$\omega : \bar{X} < -\frac{\sigma}{\sqrt{n}} Z_\alpha + \mu_0$$

Example 2

Let $X_1, X_2, ..., X_n$ be a random sample of size 20 from the bernoulli distribution

$$f(x, \theta) = \theta^x (1-\theta)^{1-x}, x = 0 \ \ or \ \ 1$$

. Obtain the most powerful size $\alpha = 0.05$ for testing

$$H_0 : \theta = 0.02 \ \ vs \ \ H_a : \theta = 0.04$$

Solution

- The likelihood function is:

$$L(X,\theta) = \prod_{i=1}^{n} \theta^{X_i}(1-\theta)^{1-X_i} = \theta^{\sum_i^n X_i}(1-\theta)^{n-\sum_{i=1}^n X_i}$$

- Under H_a

$$L(X,\theta_1) = (0.04)^{\sum_i^n X_i}(0.96)^{n-\sum_i^n X_i}$$

- Under H_0

$$L(X,\theta_0) = (0.02)^{\sum_{i=1}^n X_i}(0.98)^{n-\sum_{i=1}^n X_i}$$

- $$\frac{L(X,\theta_1)}{L(X,\theta_0)} = \frac{(0.04)^{\sum_{i=1}^n X_i}(0.96)^{n-\sum_{i=1}^n X_i}}{(0.02)^{\sum_{i=1}^n X_i}(0.98)^{n-\sum_{i=1}^n X_i}} < k$$

$$\begin{aligned}
\omega &= \{X : 2^{\sum_{i=1}^n X_i}\left(\frac{96}{98}\right)^{20-\sum_{i=1}^{20} X_i} < k\} \\
&= \{X : \sum_{i=1}^n X_i \ln 2 + (20 - \sum_{i=1}^n X_i)\ln\frac{96}{98} < \ln k\} \\
&= \sum_{i=1}^n \left(\ln 2 - \ln\frac{96}{98}\right) < \ln k - 20\ln\frac{96}{98} \qquad (5) \\
&= \{X : \sum_{i=1}^n X_i > \frac{\ln 2 - 20\ln\frac{96}{98}}{\ln\frac{1}{2} - \ln\frac{96}{98}}\} \\
&= \{X : \sum > c\}\ \textit{The rejection or critical region}
\end{aligned}$$

Cont'd

- where c satisfies $P(\sum_i^n > c|\theta = 0.02) = 0.05$
- Let $Y = \sum_{i=1}^n X_i$ then Y has the binomial distribution with parameters θ and 20
- $$f(y) = \binom{20}{y}\theta^y(1-\theta)^{20-y}$$
- So under H_0, $Y \sim binom(20, 0.02$
- $P(\sum_{i=1}^n X_i > c|\theta = 0.02) = \sum_{y=c+1}^{20}\binom{20}{y}(0.02)^y(0.98)^{20-y}$
- It is enough to determine the value of c s.t $P(\sum_{i=1}^n X_i > c|0.02) = 0.05$

Exercise 1

X is normally distributed with mean μ unknown and variance $\sigma^2 = 1$. Test the hypothesis

$$H_0 : \mu = 0 \;\; \textit{against} \;\; H_1 : \mu = 1$$

Solution to be provided in your Masomo Portal

Example 3

Suppose X is a normal random variable with mean μ_0 and unknown variance σ^2. Test the hypothesis

$$H_0 : \sigma = \sigma_0 \ \ vs \ \ H_a : \sigma = \sigma_1$$

Solution

- Let $X_1, X_2, ..., X_n$ be a random sample of size n from X.
- The likelihood function of $x_1, x_2, ..., x_n$ is

$$L(X, \sigma) = \left(\frac{1}{2\pi}\right)^{n/2}\left(\frac{1}{\sigma^2}\right)^{n/2} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu_0)^2}$$

- Since H_0 and H_1 are simple the most powerful size α test of H_0 against H_1 is given by the critical region

$$\omega = \{X : \frac{L(X, \sigma_1)}{L(X, \sigma_2)} > k\}$$

Cont'd

- Now under H_1,

$$L(X, \sigma_1) = \left(\frac{1}{2\pi}\right)^{n/2} \left(\frac{1}{\sigma_1^2}\right)^{n/2} e^{-\frac{1}{2\sigma_1^2}\sum_{i=1}^{n}(x_i - \mu_0)^2}$$

- Now under H_0,

$$L(X, \sigma_0) = \left(\frac{1}{2\pi}\right)^{n/2} \left(\frac{1}{\sigma_0^2}\right)^{n/2} e^{-\frac{1}{2\sigma_0^2}\sum_{i=1}^{n}(x_i - \mu_0)^2}$$

Cont'd

- $$\frac{L(X,\sigma_1)}{L(X,\sigma_2)} = \left(\frac{\sigma_0}{\sigma_1}\right)^n e^{-\frac{1}{2\sigma_1^2}\sum_{i=1}^n (x_i-\mu_0)^2+\frac{1}{2\sigma_0^2}\sum_{i=1}^n (x_{i-\mu_1})^2} > k$$

- Taking natural logarithm we have:

$$n\ln(\frac{\sigma_0}{\sigma_1}) - \frac{1}{2\sigma_1^2}\sum_{i=1}^n (x_i-\mu_0)^2 + \frac{1}{2\sigma_0^2}\sum_{i=1}^n (x_i-\mu_0)^2 > lnk$$

$$\frac{1}{2}\left(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}\right)\sum_{i=1}^n (X_i-\mu_0)^2 > \ln k - n\ln(\frac{\sigma_0}{\sigma_1})$$

Cont'd

- If $\sigma_1 > \sigma_0$ then $\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2} > 0$
- Then

$$\sum_{i=1}^{n}(X_i - \mu_0)^2 > \frac{2(\ln k - n\ln(\frac{\sigma_0}{\sigma_1}))}{\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}} = C_\alpha$$

- Note that:

$$X \sim N(\mu, \sigma^2), Z \sim N(0,1), Z = \frac{X - \mu}{\sigma}, Z \sim \chi_1^2$$

- Under H_0

$$X \sim N(\mu_0, \sigma_0^2)$$

Cont'd

- Where C_α satisfies $P(X \in \omega | H_0) = \alpha$
- When H_0 is true:

$$\begin{aligned}
\sum_{i=1}^{n}\left(\frac{X_i-\mu_0}{\sigma_0^2}\right)^2 = \sum_{i=1}^{n}\left(\frac{X_i-\mu_0}{\sigma_0}\right)^2 &\sim \chi_n^2 \\
&= P\left[\sum_{i=1}^{n}(X_i-\mu_0)^2 > C_\alpha | H_0\right] = \alpha \\
&= P\left[\sum_{i=1}^{n}\frac{(X_i-\mu_0)^2}{\sigma_0^2} > \frac{C_\alpha}{\sigma_0^2}\right] = \alpha \\
&= P\left[\chi_n^2 > \frac{C_\alpha}{\sigma_0^2}\right] = \alpha \\
&= \frac{C_\alpha}{\sigma_0^2} = \chi_{1-\alpha}^2 = \sigma_0^2\chi_{1-\alpha}^2
\end{aligned} \tag{6}$$

Cont'd

- The most powerful test size α test now is to reject H_0 at α level of significance whenever

$$\begin{aligned} &= \sum_{i=1}^{n}(X_i - \mu_0)^2 > \sigma_0^2\chi^2_{1-\alpha} \\ &= \frac{\sum_{i=1}^{n}(X_i - \mu_0)^2}{\sigma_0^2} > \chi^2_{1-\alpha} \end{aligned} \tag{7}$$

Cont'd

- The power of the test under H_1 is given by $P[\textit{rejecting} \;\; H_0|H_1]$:

$$
\begin{aligned}
&= P\left[\sum_{i=1}^{n}(X_i-\mu_0)^2 > \sigma_0^2\chi^2_{1-\alpha}|H_1\right] \\
&= P\left[\frac{\sum_{i=1}^{n}(X_i-\mu_0)^2}{\sigma_1^2} > \frac{\sigma_0^2\chi^2_{1-\alpha}}{\sigma_1^2}\right] \\
&= P\left[\chi^2_n > \frac{\sigma_0^2}{\sigma_1^2}\chi^2_{1-\alpha}\right]
\end{aligned}
\tag{8}
$$

Cont'd

- Let $\sigma_0^2 = 2, \sigma_1^2 = 3, n = 8, \alpha = 0.05, \sigma_0 = \sqrt{2}, \sigma_1 = \sqrt{3}$

$$
\begin{aligned}
&= P\Big[\frac{\sum_{i=1}^{n}(X_i - \mu_0)^2}{\sigma_0^2} > \frac{C_\alpha}{\sigma_0^2}\Big] \\
&= P\Big[\chi_n^2 > \frac{C_\alpha}{\sigma_0^2}\Big] = \alpha \\
&= P[\chi_n^2 > \frac{C_\alpha}{2}] = 0.05 \\
&= \frac{C_\alpha}{2} = \chi_{1-\alpha}^2 \\
&= C_\alpha = 2\chi_{1-\alpha}^2 = 2\chi_{0.05,8}^2 = 31 \\
&= \omega\{X : \sum_{i=1}^{n}(X_i - \mu_0)^2 > 31\}
\end{aligned}
\tag{9}
$$

is the critical region. If $\sigma_1 < \sigma_0$ then $\sum_{i=1}^{n}(X_i - \mu_0)^2 < C_\alpha$

Thank You!